A Cognitive Taxonomy of Numeration Systems

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Abstract

In this paper, we study the representational properties of numeration systems. We argue that numeration systems are distributed representations—representations that are distributed across the internal mind and the external environment. We analyze number representations at four levels: dimensionality, dimensional representations, bases, and symbol representations. The representational properties at these four levels determine the representational efficiencies of numeration systems and the performance of numeric tasks. From this hierarchical structure, we derive a cognitive taxonomy that can classify most numeration systems.

Introduction

We all know that Arabic numerals are more efficient than Roman numerals for calculation (e.g., 73 × 27 is easier than LXXIII × XXVII), even though they both represent the same entities—numbers. This representational effect, the effect that different representations of a common abstract structure can cause different behaviors, is a cognitive phenomenon. However, early studies of numeration systems focused on their historical and mathematical aspects (e.g., Flegg, 1983; Ifrah, 1987). Recently, the cognitive properties of numeration systems have been analyzed (e.g., Nickerson, 1988; Norman, in press; Zhang, 1992).

In this paper, we analyze number representations under different numeration systems. From this analysis, we derive a cognitive taxonomy that can classify most of the numeration systems that have been invented across the world. This taxonomy is the basis for the study of the representational effect of numeration systems.

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Dimensionality of Numeration Systems

1 D Systems

Numeration systems can be analyzed in terms of their dimensions. One of the simplest ways to represent numbers is to use stones: one stone for one, two stones for two, and so on. This Stone-Counting system has only one dimension: the quantity of stones. The Body-Counting system used by Torres islanders is another one dimensional system, where the single dimension is represented by the positions of different body parts (e.g., fingers, wrists, etc.). One dimensional systems are denoted as 1 D in this paper.

1×1 D Systems

Many numeration systems have two dimensions: one *base dimension* and one *power dimension*. The power dimension decomposes a number into hierarchical groups on a *base*. The Arabic system is a two dimensional system (Table 1) with a base dimension represented by the shapes of the ten digits (0, 1, 2, ..., 9) and a power dimension represented by positions of the digits with a base ten. For example, the middle 4 in 447 has value 4 on the base dimension and position 1 (counting from the rightmost digit, starting from zero) on the power dimension. The actual value it represents is forty (the product of its values on the base and power dimensions, i.e., 4×10^1).

The Greek system (Table 1) is a two dimensional system with special properties. Its base and power dimensions are represented by a single physical dimension—shape (a one-to-two mapping), which can only be separated as two dimensions in the mind. For example, the values on the base and power dimensions of τ (300) are 3 and 10^2, which can not be separated from the physical property of the symbol "τ". They must be separated in the mind. The separation of the single physical dimension (shape) into a base and a power dimension in the mind is required by the Greek Multiplication Algorithm, which needs to process the values on the base and power dimensions separately (see Flegg, 1983).

Two dimensional systems are denoted as 1×1 D (base×power). The structures and representations of six 1×1 D systems are shown in Table 1.

Table 1. 1×1D Numeration Systems

Systems	Example (447)	Base	Base Dimension	Power Dimension
Abstract	$\Sigma\, a_i x^i$	x	a_i	x^i
Arabic	447 $4\times10^2+4\times10^1+7\times10^0$	10	a_i = shape 0, 1, 2, ..., 9	x^i = position ... 10^2 10^1 10^0
Egyptian	𓍢𓍢𓍢𓍢∩∩∩∩||||||| $4\times10^2+4\times10^1+7\times10^0$	10	a_i = quantity The numbers of |'s, ∩'s, 𓍢's, etc.	x^i = shape | ∩ 𓍢 ... 10^0 10^1 10^2 ...
Cretan	////◐◐◐◐ʼʼʼʼʼʼʼ $4\times10^2+4\times10^1+7\times10^0$	10	a_i = quantity The numbers of ʼ's, ◐'s, /'s, etc.	x^i = shape ʼ ◐ / ... 10^0 10^1 10^2 ...
Greek	υμζ $4\times10^2+4\times10^1+7\times10^0$	10	a_i = shape α β γ ... θ 1 2 3 ... 9	x^i = shape ι κ λ ... ϙ 1×10^1 2×10^1 3×10^1...9×10^1 ρ σ τ ... ϡ 1×10^2 2×10^2 3×10^2...9×10^2
Aztec	(feather)(flag)(flag)••••••• $1\times20^2+2\times20^1+7\times20^0$	20	a_i = quantity The numbers of •'s, (flag)'s, (feather)'s, etc.	x^i = shape • (flag) (feather) ... 20^0 20^1 20^2 ...
Chinese	四百四十七 $4\times10^2+4\times10^1+7\times10^0$	10	a_i = shape 一 二 三 ... 九 1 2 3 ... 9	x^i = shape 十 百 千 ... 10^1 10^2 10^3 ...

Table 2. (1×1)×1D Numeration Systems

Systems	Example (447)	Main Base	Sub-base	Sub-base Dimension	Sub-power Dimension	Main Power Dimension
Abstract	$\Sigma\Sigma(b_{ij}y^j)x^i$	x	y	b_{ij}	y^j	x^i
Babylonian	𒐛 𒌋𒌋𒐛 $(0\times10^1+7\times10^0)60^1$ $+(2\times10^1+7\times10^0)60^0$	60	10	b_{ij} = quantity The numbers of 𒁹's and 𒌋's	y^j = shape 𒁹 = 10^0, 𒌋 = 10^1	x^i = position ... 60^2 60^1 60^0
Mayan	• •• ••̲ $(0\times5^1+1\times5^0)20^2$ $+(0\times5^1+2\times5^0)20^1$ $+(1\times5^1+2\times5^0)20^0$	20	5	b_{ij} = quantity The numbers of •'s and —'s.	y^j = shape • = 5^0, — = 5^1	x^i = position ... 20^2 20^1 20^0
Roman	CCCCXXXXVII $(0\times5^1+4\times5^0)10^2$ $+(0\times5^1+4\times5^0)10^1$ $+(1\times5^1+2\times5^0)10^0$	10	5	b_{ij} = quantity The numbers of I's, V's, X's, L's, etc.	y^j = shape I = $10^0\times5^0$ V = $10^0\times5^1$ X = $10^1\times5^0$ L = $10^1\times5^1$ ...	x^i = shape I = $5^0\times10^0$ V = $5^0\times10^1$ X = $5^0\times10^1$ V = $5^1\times10^1$ ...

(1×1)×1 D Systems

Some numeration systems have three dimensions: one *main power dimension*, one *sub-base dimension*, and one *sub-power dimension*. The sub-base and sub-power dimensions together form the *main base dimension*. Let us consider a Babylonian numeral (see Table 2):

𒐛 𒎙𒐛 $= (0\times10^1+7\times10^0)\times60^1+(2\times10^1+7\times10^0)\times60^0$ $= 7\times60^1+27\times60^0 = 420 + 27 = 447$

The main power dimension of the Babylonian system is represented by positions with a base 60. In the numeral shown above, 𒎙𒐛 (the right component) is on position 0 (60^0), and 𒐛 (the left component) is on position 1 (60^1). The value of 𒐛 (the left component) on the main base dimension is 7 ($= 0\times10^1+7\times10^0$). The actual value it represents is the product of its values on the main base and main power dimensions: $(0\times10^1+7\times10^0)\times60^1 = 420$. The main base dimension is composed of a sub-base dimension represented by quantity (the numbers of 𒁹's and 𒌋's) and a sub-power dimension represented by shape (𒁹 $= 10^0$ and 𒌋 $= 10^1$) with a base 10. For example, 𒎙𒐛 can be decomposed as $2\times10^1+7\times10^0$.

Similar to the Greek system, the sub-power and the main power dimensions of the Roman system (see Table 2) are represented by a single physical dimension—shape. They are only separable in the mind.

Three dimensional systems are denoted as (1×1)×1 D ((sub-base×sub-power)×main-power). The structures and representations of three (1×1)×1 D systems are shown in Table 2.

The Hierarchical Structure

Based on the analysis of the dimensionality of numeration systems, we can analyze number representations at four levels. Each level has an abstract structure that can be represented differently. The different representations at each level are isomorphic to each other in the sense that they all have the same abstract structure at that particular level (Figure 1).

At the *level of dimensionality*, different numeration systems can have different dimensionalities: 1D, 1×1D, (1×1)×1D, and others. However, they are all isomorphic to each other at this level in the sense that they all represent the same entities—numbers.

At the *level of dimensional representations*, isomorphic numeration systems have the same dimensionality but different dimensional representations. The physical properties used to represent the dimensions of numeration systems are usually quantity, position, and shape. For example, the base and power dimensions of 1×1D systems can be represented by shape and position (Arabic system), shape and shape (Chinese system), quantity and shape (Egyptian system), etc.

At the *level of bases*, isomorphic numeration systems have the same dimensionality, same dimensional representations, but different bases. For example, both the Egyptian and the Aztec systems are 1×1D systems, and the base and power dimensions of both systems are represented by quantity and shape. However, the base of the Egyptian system is ten while that of the Aztec system is twenty (see Table 1).

At the *level of symbol representations*, isomorphic numeration systems have the same abstract structures at the previous three levels. However, different symbols are used. For example, both the Egyptian and the Cretan systems are 1×1D systems, the two dimensions of both systems are represented by quantity and shape, and both systems have the base ten. However, in the Egyptian system, the symbols for 10^0, 10^1, and 10^2 are |, ∩, and 𓍢, while in the Cretan system, the corresponding symbols are ❛, ●, and ⁄.

The Cognitive Taxonomy

The hierarchical structure of number representations in Figure 1 is in fact a cognitive taxonomy of numeration systems that can classify numeration systems. For example, the Egyptian and Cretan systems are in the same group at the level of symbol representations; the Mayan and Babylonian systems are in the same group at the level of bases; the Arabic, Greek, Chinese, Egyptian, Cretan, and Aztec systems are in the same group at the level of dimensional representations; and all the systems in Figure 1 are in the same group at the level of dimensionality. Under this taxonomy, the lower the level at which two systems are in the same group, the more similar they are. For example, the Egyptian and the Cretan systems are more similar than the Arabic and the Babylonian systems, because the former two are in the same group

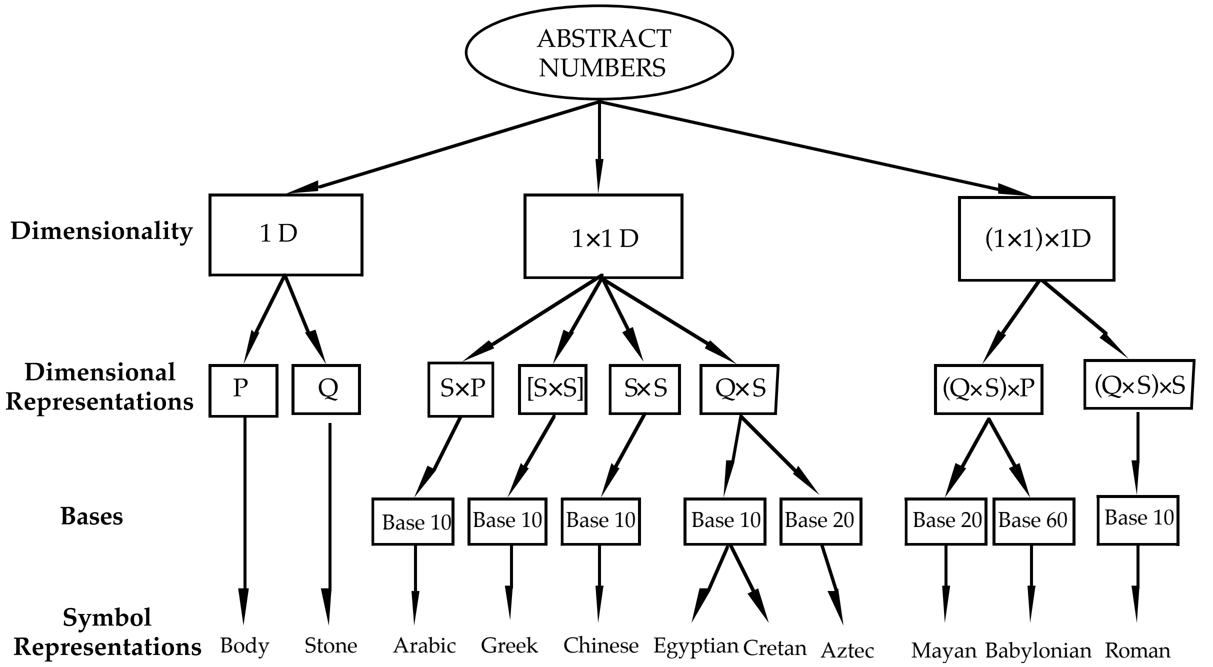

Figure 1. The hierarchical structure of number representations. At the level of dimensionality, different systems have different dimensionalities. At the level of dimensional representations, the dimensions of different systems are represented by different physical properties. P = Position, Q = Quantity, S = Shape. The two dimensions of the Greek system ([S×S]) are represented in the mind and only separable in the mind. At the level of bases, different systems may have different bases. At the level of symbol representations, different systems use different symbols.

at the level of symbol representations while the latter two at the level of dimensionality.

This cognitive taxonomy was based on the eleven systems in Tables 1 and 2. However, it can be applied to other systems as well. Let us consider a few more systems (see Ifrah, 1987, for descriptions of these systems). At the level of symbol representations, the Hebrew alphabetic system is in the group as the Greek system, and the Greek acrophonic, Dalmatian, and Etruscan systems are in the same group as the Roman system. At the level of bases, the Chinese scientific system is in the same group as the Mayan system.

In addition to written numeration systems, this taxonomy can also classify many physical systems. The following are a few examples from Ifrah (1987). The Peruvian knotted string system is a P×Q (base 10) system; the Chimpu (knotted strings used by the Indians of Peru and Bolivia) is a Q×Q (base 10) system; the knotted string system used by the German millers is a S×S (base 10) system; the Roman counting board, the Chinese abacus, and the Japanese Soroban are (Q×P)×P (main base 10 and sub-base 5) systems, and the Russian abacus is a Q×P (base 10) system.

Factors of the Representational Effect

The factors of the representational effect of numeration systems can be analyzed at the four levels of number representations.

The level of dimensionality affects the efficiency of information encoding. 1 D systems are linear, while 1×1 D and (1×1)×1 D systems are polynomial. Polynomial systems encode information more efficiently than linear systems because the number of symbols needed to encode a number in a polynomial system is proportional to the logarithm of the number of symbols needed to encode the same number in a linear system.

The level of dimensional representations is crucial for the representational effect of numeration systems. Next section analyzes the representational properties at this level in detail.

The level of bases is important for tasks involving addition and manipulation tables: the larger a base is, the larger the addition and multiplication tables and the harder they can be memorized and used.

The level of symbol representations mainly affects the reading and writing of symbols.

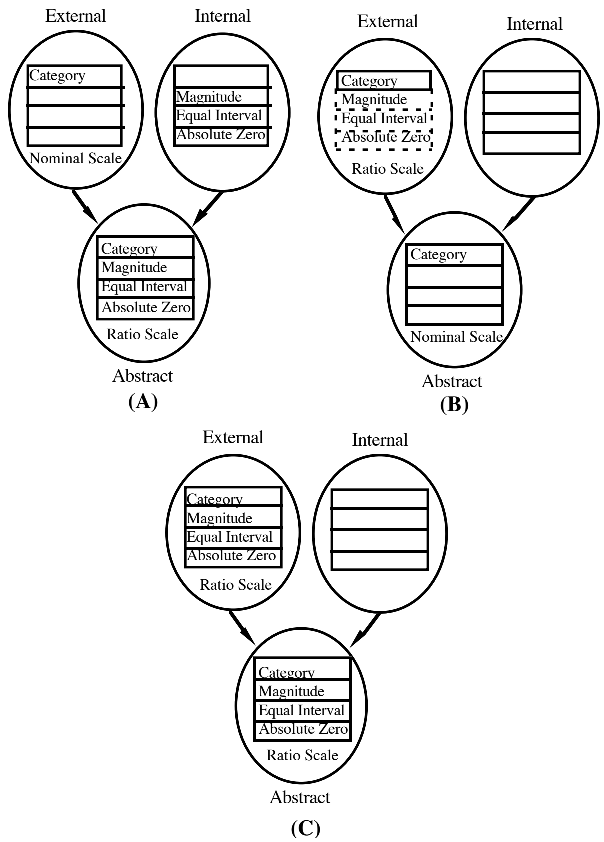

Figure 2. The distributed representation of scale information. The abstract representational space can be decomposed into an internal and an external representational space. (A) A nominal scale represents a ratio scale. The extra information of the ratio scale either has to be represented in the internal representational space or not represented at all. (B) A ratio scale represents a nominal scale. The extra information of the ratio scale may cause misperception on the nominal scale. (C) A ratio scale represents a ratio scale. This is an efficient and accurate representation.

Dimensional Representations

Dimensional representations of numeration systems can dramatically affect the performance of numeric tasks. In this section, we analyze the distributed representation of scale information and the separability of dimensions.

The Distributed Representation of Scale Information

Every dimension is on a certain type of scale, which is the abstract measurement property of the dimension. Stevens (1946) identified four major types of psychological scales: ratio, interval, ordinal, and nominal. Each type has one or more of the following properties (see Table 3): category, magnitude, equal interval, and absolute zero. *Category* refers to the property that the instances on a scale can be distinguished from each another. *Magnitude* refers to the property that one instance on a scale can be judged greater than, less than, or equal to another instance on the same scale. *Equal interval* refers to the property that the magnitude of an instance represented by a unit on the scale is the same regardless of where on the scale the unit falls. An *absolute zero* is a value which indicates that nothing at all of the property being represented exists.

Table 3. Properties of Psychological Scales

	ratio	interval	ordinal	nominal
category	yes	yes	yes	yes
magnitude	yes	yes	yes	no
equal internal	yes	yes	no	no
absolute zero	yes	no	no	no
example	quantity	time	softness	shape

From Table 3 we can see that the four types of scales have an order of representational power: ratio > interval > ordinal > nominal. A higher scale possesses more information than a lower scale. The scale information of a dimension can be distributed across internal and external representations (Figure 2; see Zhang, 1992; Zhang & Norman, forthcoming). When a higher dimension is represented by a lower dimension (e.g., a ratio dimension represented by a nominal dimension), the extra information of the higher dimension either has to be represented internally, or not represented at all, because it is not embedded in the physical properties of the lower dimension (Figure 2A). When a lower dimension is represented by a higher dimension (e.g., a nominal dimension represented by a ratio dimension), the extra information of the higher dimension may cause misperception on the lower dimension (Norman, in press; Figure 2B). Thus, in order for a representation to be efficient and accurate, the scale types of the represented and the representing dimensions should match (Figure 2C).

Quantity, position, and shape are the physical dimensions used in most numeration systems. The dimensions of all 1 D systems and the base and power dimensions of all multidimensional systems are on ratio scales. These ratio dimensions are represented externally by quantity (ratio) and internally by shape (nominal), and externally by ratio position (as in the Arabic system) and internally by ordinal position (as in the Body-Counting system). For example, for the Arabic system, the power dimension is represented externally by position, and the base dimension is represented internally by shape. In contrast, for the Egyptian system, the power dimension is represented internally by shape, and the base dimension is represented externally by quantity.

The Separability of Dimensions

Another major factor of dimensional representations is whether the dimensions of a multi-dimensional numeration system is externally separable (see Garner, 1974, for a general discussion on separable and integral dimensions). For example, the shape (base dimension) and position (power dimension) of each digit in an Arabic numeral are externally separable (by perceptual processes). For the Greek system, however, the base and power dimensions are represented by a single physical dimension (shape). They are only separable in the mind with the participation of high-level cognitive processes.

Numeric Tasks

Whether the dimensions of numeration systems are represented internally or externally and whether they are externally separable can greatly affect the difficulty of numeric tasks.

Zhang (1992) analyzed in detail the relation between the format of dimensional representations and the difficulty of numeric tasks. For example, the basic component in the polynomial multiplication method for 1×1 D systems is the multiplication of individual terms ($a_i x^i \times b_j y^j$), which has five basic steps:

(1) Get powers of $a_i x^i$ and $b_j y^j$: i, j;
(2) Add powers: $i + j = p_{ij}$;
(3) Get base values of $a_i x^i$ and $b_j y^j$: a_i , b_j;
(4) Multiply base values: $a_i \times b_j = b_{ij}$;
(5) Attach powers to the product of base values: $a_i x^i \times b_j y^j = b_{ij} \times 10^{p_{ij}}$.

Whether the information needed to execute a step is internal or external is jointly determined by whether the base and power dimensions are externally separable and whether a dimension is represented externally or internally. Internal information needs more cognitive processing than external information. If the base and power dimensions are not externally separable, then all the five steps are internal (e.g., the Greek system). For externally separable base and power dimensions, if the power dimension is represented externally then Steps 1, 2, and 5 are external, and if the base dimension is represented externally then Step 3 is external. Step 4 is usually internal because the multiplication table is usually memorized. For the Arabic system, Steps 1, 2, and 4 are external and Steps 3 is internal because its power dimension is external and its base dimension is internal. Step 4 is also internal if an internal multiplication table is used. From this analysis we can see that the Arabic system is more efficient than the Greek system for multiplication, because the former has three external steps and the latter has no external steps.

Conclusion

Numbers are represented hierarchically at four levels, the representational properties at which determine the efficiencies of numeration systems. This hierarchical structure is also a cognitive taxonomy of numeration systems, which can classify most numeration systems.

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